STAT 2593

Lecture 019 - The Exponential and Gamma Distributions

Dylan Spicker

The Exponential and Gamma Distributions

Learning Objectives

1. Understand the exponential distribution and its properties.
2. Understand the relationship between the exponential and poisson distributions.
3. Understand the gamma distribution, and the important iterations of it.
4. Understand the gamma function, and its properties.


USERS: LAST 7 DAYS USING MEDIAN
LOAD TIME VS BOUNCE RATE
OPTIONS
Median Page Load (LUX): 2.056s
Bounce Rate
7s
57.1%
Page Load (LUX)
Bounce Rate
START RENDER VS BOUNCE RATE
OPTIONS
Median Start Render (LUX): 1.031s
Start Render (LUX)
Bounce Rate
PAGE VIEWS VS ONLOAD
OPTIONS
Page Load (LUX)
0.7s
Page Views (LUX)
2.7Mpvs
Bounce Rate (LUX)
40.6%
SESSIONS
OPTIONS
Sessions (LUX)
479K
Session Length (LUX)
17min
PVs Per Session (LUX)
2pvs

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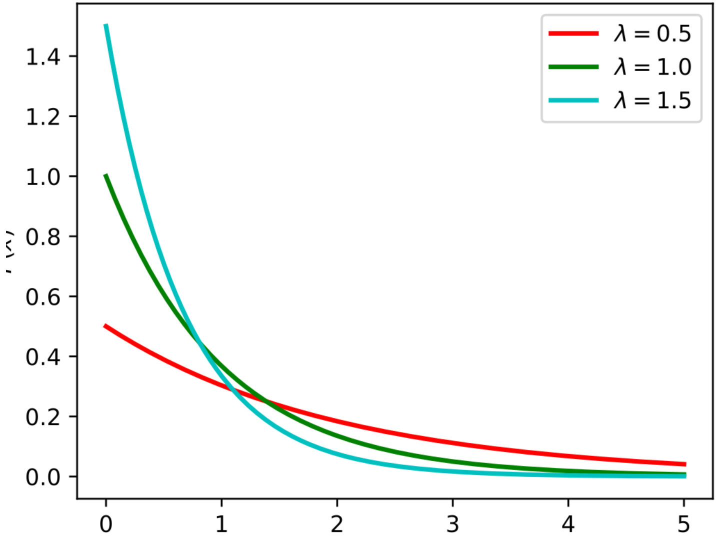
  - The CDF is given by $F(x) = 1 - \exp(-\lambda x)$.

The Exponential Distribtuion, Visually



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  - The time between successive events follows an $\text{Exp}(\alpha)$ distribution.

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  - Both parameters are strictly positive.
  - We have that $E[X] = \alpha\beta$ and that $\text{var}(X) = \alpha\beta^2$.
- $\Gamma(\alpha)$ is called the **gamma function** and is defined as

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx.$$

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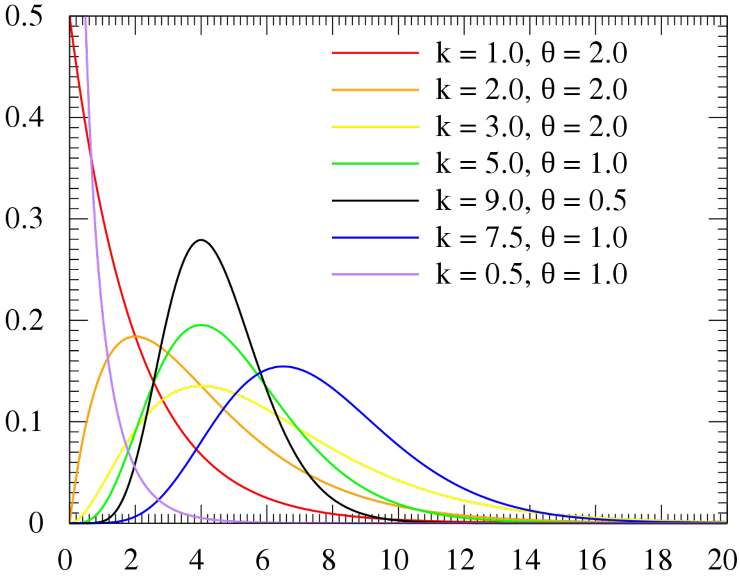

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  - The CDF of the Gamma is given by

$$F_X(x) = \frac{1}{\Gamma(\alpha)} \gamma\left(\alpha, \frac{x}{\beta}\right).$$

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  - The chi-square is an important distribution in statistics.
  - We will often write $X \sim \chi^2_\nu$.
  - If $\nu = 1$, then the chi-square distribution is the distribution of Z^2, where $Z \sim N(0, 1)$.

Summary

- The exponential distribution is a useful distribution for characterizing skewed data.
- The exponential distribution has closed form PDF, CDF, expectation, and variance.
- The exponential distribution characterizes the between-event times in a poisson process.
- The gamma distribution is a two-parameter distribution which generalizes the exponential.
- The chi-square distribution is a specific case of the gamma distribution.